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DAMAGE FIELD NEAR A STATIONARY CRACK TIP

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Abstract—In this paper the plastic damage field near a stationary crack tip is investigated. Power law strain softening is taken as the damage model. It is found that the damage field is localized in certain sectors at the crack tip. The stress and strain behavior near the crack tip is revealed analytically.

1. INTRODUCTION

The stress and strain distribution near a crack tip is an important problem in fracture mechanics. For a stationary crack, the tip field is relatively more simple than that for a growing crack. If the material is linear elastic, the stress and strain possess the $r^{-1/2}$ singularity at the crack tip. For a power-law hardening plastic material, the crack-tip field possesses the so-called HRR singularity (Hutchinson, 1968; Rice and Rosengren, 1968), i.e.

$$\varepsilon \approx r^{-n/(n+1)}, \quad \sigma \approx r^{-1/(n+1)}, \tag{1}$$

where r is the distance to the crack tip and n is the hardening exponent, namely $\varepsilon \approx \sigma^n$. For a perfectly plastic material, the stationary crack-tip field was investigated by Gao (1980a,b) and the general solution containing a free parameter was obtained. The same problem was studied by Dong and Pan (1990) and numerical works were supplemented. But some engineering materials possess other features during deformation, i.e. strain softening. For this kind of material, Gao (1986) gave an analysis of a propagating crack-tip field, but it is not valid for a stationary crack because it often possesses quite different features.

In particular, in the moving coordinate system attached to the propagating crack in perfect plasticity, elastic unloading may result in the appearance of the wake plastic zone. For a viscoplastic power law, the moving crack has been investigated by Kachanov (1978) and Hui and Riedel (1981), who showed a singular stress field similar to eqn (1) which does not depend upon a free parameter like the stress intensity factor K^σ of the HRR field for a stationary crack. For the viscoplastic singularity of the moving crack, Bui (1993) has coined the term "Soliton" because of the similarity between the Kachanov equation and the Korteweg and de Vries equation for the nonlinear wave called "soliton", whose solution is entirely determined by the propagation velocity $\dot{a}$. He considered that the fracture criterion is given by the viscoplastic "soliton" itself, so that once the inner viscoplastic singularity field and the outer elastic stress far field of the form $\sigma \simeq K_I g_I(\theta) r^{-1/2}$ governed by K_I have been matched, the macroscopic criterion $f(K_I, \dot{a}) = 0$ will follow.

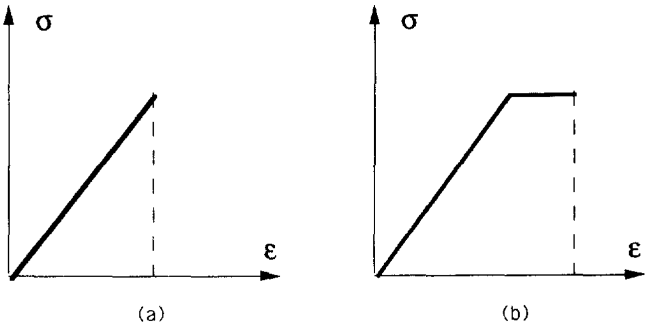

Fig. 1. Material behavior.

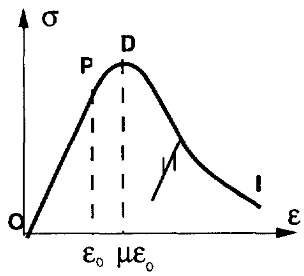

Fig. 2. Plastic damage.

In damage theory, Bui and Ehrlacher (1980) gave an analytical solution to the damage field of a crack based on the material model as shown in Figs 1(a) and (b).

In this paper we will consider the stationary crack-tip field in the strain-softening material shown in Fig. 2. The special material character shown in Figs 2 is that after peak point D, the stress decreases with increasing strain. Therefore, in block DI, the material becomes unstable. A very interesting question is: can an unstable material domain exist in the stationary crack-tip field? The result of this paper shows that unstable materials do exist in certain domains. This can be explained by the fact that the damage localization is constrained for the plane strain case by the far-field elastic zone surrounding the crack tip. As for the stability of the solution obtained, we still cannot give any assertion with our analysis. But we can presume that among the possible solutions corresponding to different parameters, at least one is stable. All the results for crack-tip fields obtained from continuum mechanics possess the common feature that they do not provide enough information to predict the crack behavior without a fracture criterion. Therefore, the analysis of the present paper still has no direct application without the formulation of a reasonable micromechanical model of fracture. But a knowledge of a macro-field near a crack tip is necessary for any fracture analysis.

2. BASIC EQUATIONS

For convenience we only consider a mode I crack under plane strain conditions. The polar coordinates (r, θ) are as shown in Fig. 3. Let $\sigma_{\alpha\beta}$ $(\alpha = r, \theta;\ \beta = r, \theta)$ denote the stress in the plane. We assume that the material is incompressible; then from the condition $\varepsilon_z = 0$ we have

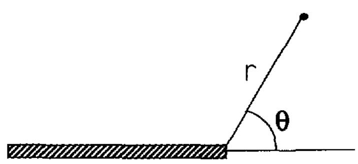

Fig. 3. Coordinates.

$$\sigma_{zz} = \tfrac{1}{2}(\sigma_{rr}+\sigma_{\theta\theta}), \tag{2}$$

therefore the effective stress σ is given by

$$\sigma = \sqrt{3}[\tfrac{1}{4}(\sigma_{rr}-\sigma_{\theta\theta})^2+\sigma_{r\theta}{}^2]^{1/2}. \tag{3}$$

In order to give the constitutive relationship, we consider the uniaxial tension. We assume that the tensile stress σ is given by the formula

$$\sigma = \begin{cases} E\varepsilon, & \varepsilon < \varepsilon_0 \\ E\varepsilon\left[\left(\dfrac{n\varepsilon}{\varepsilon_0}+\mu\right)\Big/(n+\mu)\right]^{-1-1/n}, & \varepsilon \geqslant \varepsilon_0 \end{cases} \tag{4}$$

where E is the Young's modulus and $\mu > 1$. The ε–σ curve is shown in Fig. 2. OP is called the elastic block. PD is called the plastic hardening block and DI is called the plastic softening block, i.e. the damage block, ε_0 is the yield strain, and $\mu\varepsilon_0$ is the strain corresponding to stress peak. According to eqns (4) we can obtain the plastic potential

$$F = \sigma-(\sigma+E\varepsilon_P)\left(1+\frac{\mu}{n}\right)^{1+1/n}\left(\frac{\varepsilon_P}{\varepsilon_0}+\frac{\sigma}{E\varepsilon_0}+\frac{\mu}{n}\right)^{-1-1/n} \tag{5}$$

in which $\varepsilon_P = \varepsilon-\sigma/E$. Equation (5) is also valid for the plane strain case provided σ is given by eqn (3) and ε_P is given by

$$\varepsilon_P = \frac{2}{\sqrt{3}}\int[\tfrac{1}{4}(\dot{\varepsilon}_{rr}-\dot{\varepsilon}_{\theta\theta})^2+\dot{\varepsilon}_{r\theta}^2]^{1/2}\,\mathrm{d}t. \tag{6}$$

Let $\varepsilon_{\alpha\beta}$ denote the strain; the two-dimensional constitutive relationship can be written as

$$\dot{\varepsilon}_{\alpha\beta} = \frac{3}{2E}\dot{S}_{\alpha\beta}+\lambda\frac{\partial F}{\partial\sigma}\cdot\frac{3}{2\sigma}S_{\alpha\beta}, \tag{7}$$

where

$$S_{\alpha\beta} = \sigma_{\alpha\beta}-\tfrac{1}{2}\sigma_{\gamma\gamma}\delta_{\alpha\beta} \tag{8}$$

and the superdot is the time derivative. The flow factor λ can be determined by eqn (5):

$$\lambda = -\dot{\sigma}\left(\frac{\partial F}{\partial\varepsilon_{\mathrm{p}}}\right)^{-1}. \tag{9}$$

Now, the attention is focused on the crack-tip field where the plastic strain ε_{p} is very large so that eqn (5) can be simplified as

$$F = \sigma-C\varepsilon_{\mathrm{p}}^{-1/n}, \tag{10}$$

where $C = E[(1+(\mu/n))\varepsilon_0]^{1+1/n}$. From eqns (9) and (10) we obtain

$$\lambda = -\dot{\sigma}\frac{n}{C}\left(\frac{C}{\sigma}\right)^{n+1};\tag{11}$$

then eqn (7) can be written as

$$\dot{\varepsilon}_{\alpha\beta} = \frac{3}{2E}\dot{S}_{\alpha\beta} - d\dot{\sigma}\sigma^{-n-2}S_{\alpha\beta},\tag{12}$$

in which $d = \frac{3}{2}nC^n$. For the stationary crack-tip field, we can introduce the stress function ϕ so that the equilibrium equation can be satisfied automatically

$$\begin{cases}\sigma_{rr} = \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2} + \dfrac{1\partial\phi}{r\partial r}, & \sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2},\\ \sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left(\dfrac{\partial\phi}{r\partial\theta}\right). & \end{cases}\tag{13}$$

Besides, the strain must satisfy the compatibility equation

$$\Delta(\dot{\varepsilon}_{rr} + \dot{\varepsilon}_{\theta\theta}) + \left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\partial^2}{\partial r^2} - \frac{3\partial}{r\partial r}\right)(\dot{\varepsilon}_{rr} - \dot{\varepsilon}_{\theta\theta}) - \frac{4}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial\theta}\dot{\varepsilon}_{r\theta}\right) = 0.\tag{14}$$

Equations (3) and (12)–(14) are the basic equations for a stationary crack-tip field.

3. ASYMPTOTIC EQUATIONS

In order to solve the basic equations (12)–(14), we consider a special case of the field such that the scale of the plastic domain is small so that it can be treated as a self-similar field. Let t denote the time parameter. We introduce a new variable

$$\rho = \frac{r}{t}.\tag{15}$$

We assume that the stress and strain are functions of ρ and θ. Then we have

$$\phi = r^2\Phi(\rho,\theta).\tag{16}$$

From eqns (13) and (16) we obtain

$$\begin{cases}\sigma_{rr} = 2\Phi + \dfrac{\partial^2\Phi}{\partial\theta^2} + \rho\dfrac{\partial\Phi}{\partial\rho}, & \sigma_{\theta\theta} = 2\Phi + 4\rho\dfrac{\partial\Phi}{\partial\rho} + \rho^2\dfrac{\partial^2\Phi}{\partial\rho^2},\\ \sigma_{r\theta} = -\dfrac{\partial\Phi}{\partial\theta} - \rho\dfrac{\partial^2\Phi}{\partial\rho\,\partial\theta}. & \end{cases}\tag{17}$$

Noting that $\mathrm{d}/\mathrm{d}t = -(\rho/t)\partial/\partial\rho$, then from eqn (17) we have

$$\begin{cases} \dot{\sigma}_{rr} = -\dfrac{\rho}{t}\left(3\dfrac{\partial\Phi}{\partial\rho}+\rho\dfrac{\partial^2\Phi}{\partial\rho^2}+\dfrac{\partial^3\Phi}{\partial\rho\,\partial\theta^2}\right), \\ \dot{\sigma}_{\theta\theta} = -\dfrac{\rho}{t}\left(6\dfrac{\partial\Phi}{\partial\rho}+6\rho\dfrac{\partial^2\Phi}{\partial\rho^2}+\rho^2\dfrac{\partial^3\Phi}{\partial\rho^3}\right). \\ \dot{\sigma}_{r\theta} = \dfrac{\rho}{t}\left(2\dfrac{\partial^2\Phi}{\partial\rho\,\partial\theta}+\rho\dfrac{\partial^3\Phi}{\partial\rho^2\,\partial\theta}\right). \end{cases} \tag{18}$$

Further let

$$\Phi = \rho^{\delta} f(\theta). \tag{19}$$

By substituting eqn (19) into eqn (18), it follows that

$$\begin{cases} \sigma_{rr} = \rho^{\delta}[f''+(2+\delta)f], \quad \sigma_{\theta\theta} = \rho^{\delta}(1+\delta)(2+\delta)f, \\ \sigma_{r\theta} = -\rho^{\delta}(1+\delta)f', \end{cases} \tag{20}$$

$$\dot{\sigma}_{\alpha\beta} = -\frac{\delta}{t}\sigma_{\alpha\beta}, \quad \dot{\sigma} = -\frac{\delta}{t}\sigma. \tag{21}$$

Substituting eqns (20) and (21) into eqn (12) and neglecting the subordinate terms, we obtain

$$\varepsilon_{\alpha\beta} = \frac{d}{n}S_{\alpha\beta}/\sigma^{n+1}. \tag{22}$$

Using eqns (14), (22) and (3), after a long derivation we obtain

$$f''''\left[1-(n+1)\frac{S^2}{k^2}\right]+\left[4(1+\delta)(1-n\delta)-\delta(2+\delta)+n\delta(2-n\delta)+(n+1)\delta(2+\delta)\frac{S^2}{k^2}\right]f''$$
$$-n\delta^2(2-n\delta)(2+\delta)f+2(n+1)(n+3)S\frac{(SS'+TT')^2}{k^4}-2(n+1)\frac{S}{k^2}(TT''+S'^2+T'^2)$$
$$-2(n+1)\frac{SS'+TT'}{k^2}\{f'''-[\delta(2+\delta)-2(1+\delta)(1-n\delta)]f'\} = 0 \tag{23}$$

in which

$$\begin{cases} T = -(1+\delta)f', \\ S = \frac{1}{2}[f''-\delta(2+\delta)f], \\ k = (S^2+T^2)^{1/2}. \end{cases} \tag{24}$$

Equation (23) is the governing equation of the damage field. If there exists an elastic domain in the crack-tip field, the compatibility equation should be

$$f''''+[\delta^2+(2+\delta)^2]f''+\delta^2(2+\delta)^2 f = 0. \tag{25}$$

For a mode I crack, f is an even function of θ. At the crack surface we have

$$f(\pi) = f'(\pi) = 0. \tag{26}$$

Equation (26) is the boundary condition of eqns (23) or (25).

4. ASSEMBLY OF CRACK-TIP FIELD

In order to obtain the whole solution to the crack-tip field we must presume various structures of the field. Using trial and error, we find a reasonable structure for the field as shown in Fig. 4.(a), (b). In Fig. 4(b), $\mathscr{E}$ and $\mathscr{E}'$ are elastic sectors governed by eqn (25). $\mathscr{D}$ is the damage sector that is governed by eqn (23). Γ_1 and Γ_2 are the boundaries of damage sector $\mathscr{D}$. Let f_1 be the solution to eqn (25) in sector $\mathscr{E}$; considering f_1 as an even function, we have

$$f_1 = a_1 \cos\delta\theta + a_2 \cos(2+\delta)\theta. \tag{27}$$

Let f_2 be the solution to eqn (25) in sector $\mathscr{E}'$; considering the boundary condition (26)

$$f_2 = b_1[\cos\delta(\theta-\pi) - \cos(2+\delta)(\theta-\pi)] + b_2\left[\sin\delta(\theta-\pi) - \frac{\delta}{2+\delta}\sin(2+\delta)(\theta-\pi)\right], \tag{28}$$

where a_1, a_2, b_1, b_2 are constants to be determined from the continuity conditions of neighboring sectors. Since the stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ must be continuous across Γ_1 and Γ_2, then

$$\begin{cases} f_1(\alpha) = f(\alpha), & f_1'(\alpha) = f'(\alpha), \\ f_2(\pi-\beta) = f(\pi-\beta), & f_2'(\pi-\beta) = f'(\pi-\beta), \end{cases} \tag{29}$$

where f is only for sector $\mathscr{D}$.

Furthermore, it is required that the displacements should be continuous across Γ_1 and Γ_2. For incompressible material these conditions are equivalent to the following (Gao, 1980b):

$$\begin{cases} [\varepsilon_{rr}]\Gamma_i = 0, \\ \left[\dfrac{\partial\varepsilon_{rr}}{\partial\theta}\right]\Gamma_i - 2[\varepsilon_{r\theta}]\Gamma_i - 2r\dfrac{\mathrm{d}}{\mathrm{d}r}[\varepsilon_{r\theta}]\Gamma_i = 0, \quad i = 1, 2, \end{cases} \tag{30}$$

where $[\cdot]\Gamma_i$ denotes the jump of a quantity across Γ_i. According to eqn (7), in sectors $\mathscr{E}$ and $\mathscr{E}'$ we have

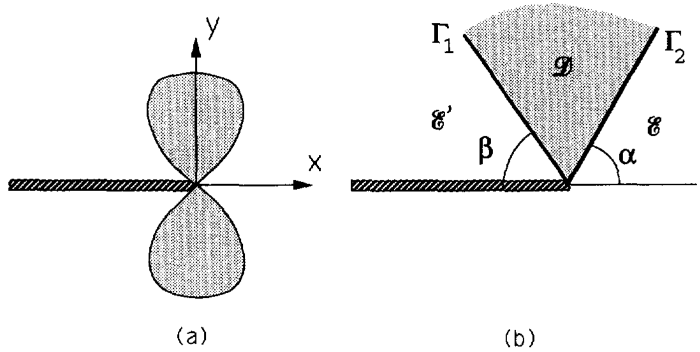

Fig. 4. Crack-tip field.

$$\varepsilon_{\alpha\beta} = \frac{3}{2E} S_{\alpha\beta}. \tag{31}$$

But for sector $\mathscr{D}$, the strain is given by eqn (22). Comparing eqns (22) and (31) and noting that $S_{\alpha\beta} \approx r^{\delta}$, $\sigma \approx r^{\delta}$, we can conclude that in order to satisfy eqn (30) it is required that

$$\begin{cases} \varepsilon_{rr} = 0, \\ \dfrac{\partial \varepsilon_{rr}}{\partial \theta} - 2\varepsilon_{r\theta} - 2r\dfrac{\partial}{\partial r}\varepsilon_{r\theta} = 0, \quad \text{when } \theta = \alpha_{+0}, \pi - \beta_{-0}. \end{cases} \tag{32}$$

From eqns (32), (22) and (20) we obtain

$$\begin{cases} f'' - \delta(2+\delta)f = 0, \\ f''' + [4(1+\delta)(1-n\delta) - \delta(2+\delta)]f' = 0, \quad \text{when } \theta = \alpha, \pi - \beta. \end{cases} \tag{33}$$

Equation (33) gives four boundary conditions for eqn (23) in $\mathscr{D}$, and eqn (29) gives four conditions for a_1, a_2, b_1, b_2. Finally, we should give the conditions to determine the angles α, β of Γ_1 and Γ_2. From the point of view of mechanics, we require that the solution possesses as few discontinuities as possible (minimum dissipation of energy); then we specify that, at Γ_1 in sector $\mathscr{E}$ and at Γ_2 in sector $\mathscr{E}'$, $\varepsilon_{rr} = 0$ is also true, i.e.

$$\begin{cases} f_1''(\alpha) - \delta(2+\delta)f_1(\alpha) = 0, \\ f_2''(\pi-\beta) - \delta(2+\delta)f_2(\pi-\beta) = 0. \end{cases} \tag{34}$$

Equations (29), (33) and (34) provide the conditions to determine the four-order equation (23) with six constants a_1, a_2, b_1, b_2, α, β. As for the eigenvalue δ, it must be determined from the solution. A numerical solution of eqn (23) has been tried. Since eqns (23) and (25) are homogeneous, we can specify $f_1(0) = 1$. Furthermore, we fix a proper value of α, then using the first of eqns (34) we obtain

$$a_1 = \frac{(2+\delta)\cos(2+\delta)\alpha}{(2+\delta)\cos(2+\delta)\alpha - \delta\cos\delta\alpha}, \quad a_2 = 1 - a_1. \tag{35}$$

Thus, using eqns (29) and (33), we can start the calculation of eqn (23) from Γ_1 to Γ_2 and examine all the conditions of eqns (29), (33) and (34). The calculation shows that for a good value of δ (i.e. all the conditions are nearly satisfied), the boundary Γ_2 is very difficult to determine and $\delta \approx 1/n$.

5. ANALYTICAL SOLUTION

From hints in the numerical work, we now try to analytically solve eqn (23). Let

$$\delta = 1/n \tag{36}$$

and force

$$S = 0. \tag{37}$$

Then eqns (23) and (33) become

$$f'''' + [1-\delta(2+\delta)]f'' - \delta(2+\delta)f = 0, \tag{38}$$

$$f'' - \delta(2+\delta)f = 0 \text{ and } f''' - \delta(2+\delta)f' = 0 \quad \text{at } \theta = \alpha, \pi-\beta. \tag{39}$$

Evidently, if we take

$$\begin{cases} f = C_1 \cosh\lambda(\theta-\alpha) + C_2 \sinh\lambda(\theta-\alpha), \\ \lambda = [\delta(2+\delta)]^{1/2} = \dfrac{1}{n}(2n+1)^{1/2}, \end{cases} \tag{40}$$

then eqn (38) and boundary conditions (39) are satisfied. Furthermore, we can see that the forced condition (37) is also satisfied by eqn (40). Therefore, the analytical solution to eqn (23) is found. Now, we can complete the whole solution to the crack tip. To begin with, we specify an arbitrary α, then from eqn (35) we can obtain a_1 and a_2, using the first of eqns (29) and (40), obtaining

$$\begin{cases} C_1 = \dfrac{1}{\Delta}[\cos^2(1+\delta)\alpha + \cos 2\alpha], \\ C_2 = \dfrac{\delta(2+\delta)}{\lambda\Delta}\sin 2\alpha, \\ \Delta = (2+\delta)\cos(2+\delta)\alpha - \delta\cos\delta\alpha \end{cases} \tag{41}$$

Substituting eqns (40), (41) and (28) into the last two equations of (29) and the second of eqn (34), and further eliminating b_1 and b_2, we obtain

$$\begin{aligned} 2\lambda(1+\delta)\sin^2\beta[2\cos\delta\alpha\cos(2+\delta)\alpha\cosh\lambda(\pi-\beta-\alpha) \\ +\lambda\sin 2\alpha\sinh\lambda(\pi-\beta-\alpha)] + [(1+\delta)\sin 2\beta - \sin 2(1+\delta)\beta] \\ \times[2\cos\delta\alpha\cos(2+\delta)\alpha\sinh\lambda(\pi-\beta-\alpha) + \lambda\sin 2\alpha\cosh\lambda(\pi-\beta-\alpha)] = 0. \end{aligned} \tag{42}$$

Equation (42) gives the relationship between α and β. The permitted ranges of α, β for a given n are shown in Fig. 5. Therefore the crack-tip field contains a free parameter (α or

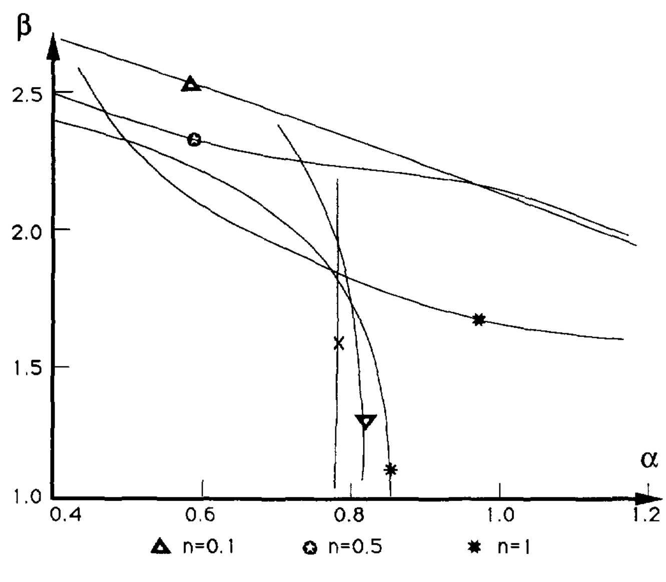

Fig. 5. Relationships of α and β for different values of n.

β). When $n \to \infty$ the solution is consistent with that for the perfectly plastic case (Gao, 1980a), namely $\alpha = \pi/4$, but β is uncertain.

6. REMARKS

An interesting problem is the crack-tip field in an elastic material that possesses a similar stress–strain curve to that shown in Fig. 2. Letting $e_{\alpha\beta}$ denote the strain deviator for incompressible material, we have $e_{\alpha\beta} = \varepsilon_{\alpha\beta}$.

The effective strain ε is defined as

$$\varepsilon = (\tfrac{2}{3} e_{\alpha\beta} \cdot e_{\alpha\beta})^{1/2}. \tag{43}$$

If we assume that the strain energy density is a function of ε, $U(\varepsilon)$, then the stress $\sigma_{\alpha\beta}$ and its deviator $S_{\alpha\beta}$ are

$$\sigma_{\alpha\beta} = S_{\alpha\beta} + p\delta_{\alpha\beta}, \quad S_{\alpha\beta} = \frac{\mathrm{d}U}{\mathrm{d}\varepsilon} \cdot \frac{2}{3\varepsilon} e_{\alpha\beta}, \tag{44}$$

where $\delta_{\alpha\beta}$ is the unit tensor and p is the undetermined hydro-stress. If, for large enough ε, $U = nA\varepsilon^{(n-1)/n}/(n-1)$, then the effective stress is $\sigma = A\varepsilon^{-1/n}$. For this case, the crack-tip field given in this paper is also valid.

7. CONCLUSIONS

The strain damage field problem near a stationary crack tip in a plastic material is solved. The damage zone is localized in certain sectors. The angle of the damage sector is a free parameter that cannot be determined from the asymptotic solution. The stress tends to zero when the crack tip is approached. The solution obtained for a plastic softening material is also valid for similar elastic softening materials.

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